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# Categorical entropy and Periodic points

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# Categorical entropy and Periodic points

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## 1. Introduction

Let  $X$  be a quasi-projective variety over a field  $k$  and  $f$  a finite endomorphism of  $X$ . Then, we can define a (algebraic) dynamical system  $(X, f)$ . By **categorification of the dynamics**:  $(X, f) \rightsquigarrow (\text{Perf}(X), \mathbb{L}f^*)$ , where  $\text{Perf}(X)$  is the triangulated category of perfect complexes on  $X$  and  $\mathbb{L}f^*$  is the derived pull-back of  $f$ , we estimate the entropy of categorical dynamics motivated by the Gromov-Yomdin's fundamental theorem, and apply the categorical entropy to periodic points of  $f$ .

The entropy of categorical dynamics is also related to the “dimension” of triangulated categories and the mass growth of Bridgeland stability conditions.

## 2. Entropy of categorical dynamics

Let  $\Phi$  be a Fourier-Mukai type endofunctor of  $\text{Perf}(X)$  and  $G$  a split-generator of  $\text{Perf}(X)$ . For a very ample line bundle  $\mathcal{L}$  on  $X$ ,  $\bigoplus_{i=0}^{\dim X} \mathcal{L}^i$  is a fundamental example of split-generators of  $\text{Perf}(X)$ .

**Definition 1.** The complexity  $\delta(G, M)$  of  $M \in \text{Perf}(X)$  with respect to  $G$  is defined as follows:

$$\delta(G, M) := \min \left\{ p \in \mathbb{Z}_{>0} \mid \begin{array}{ccccccc} 0 & \xrightarrow{\quad} & A_1 & \cdots & A_{p-1} & \xrightarrow{\quad} & A_p \oplus M \\ & \searrow & \downarrow & & \downarrow & \searrow & \\ & & G[n_1] & \cdots & G[n_p] & & \end{array} \right\} \in \mathbb{Z}_{>0}.$$

**Definition 2** (Dimitrov–Haiden–Katzarkov–Kontsevich [1]). **The entropy  $h(\Phi)$**  of categorical dynamics  $(\text{Perf}(X), \Phi)$  is defined as follows:

$$h(\Phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \delta(G, \Phi^n G) \in \mathbb{R}_{\geq 0}.$$

The entropy of categorical dynamics is similar to **the topological entropy  $h_{\text{top}}(f)$**  in the sense of sharing many standard properties:

**Proposition 3.** We have the followings.

- (i)  $h(\Phi^l) = lh(\Phi)$ .
- (ii)  $\Phi_1 \Phi_2 = \Phi_2 \Phi_1 \Rightarrow h(\Phi_1 \Phi_2) \leq h(\Phi_1) + h(\Phi_2)$ .
- (iii) The entropy  $h(F)$  is invariant in conjugacy classes of  $\text{Auteq}(\text{Perf}(X))$ .

The above definitions and properties hold for more general triangulated categories. Especially, (iii) is generalized as an inequality for semi-conjugate categorical dynamics. In smooth projective cases, we have the following theorem:

**Theorem 4** (DHKK [1]). When  $X$  is smooth projective over  $k$ , the following holds:

$$h(\Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{m \in \mathbb{Z}} \dim_k \text{Hom}_{\text{Perf}(X)}(G, \Phi^n G[m]) \right).$$

This theorem enables us to compute many examples easily.

## 3. Gromov-Yomdin type equality

In this section, suppose that  $X$  is smooth projective varieties over  $k = \mathbb{C}$ .

**Theorem 5** (K-Takahashi [4], Ouchi [7]). We have the following:

$$h(\mathbb{L}f^*) = \log \rho(\mathcal{N}(\mathbb{L}f^*)),$$

where  $\rho(\mathcal{N}(\mathbb{L}f^*))$  is the spectral radius of  $\mathcal{N}(\mathbb{L}f^*)$  on  $\mathcal{N}(\text{Perf}(X))$ .

**Corollary 6.** We have the following:

$$h_{\text{top}}(f) = h(\mathbb{L}f^*).$$

Motivated by the Gromov-Yomdin theorem:  $h_{\text{top}}(f) = \log \rho(N(f))$  and Thm5, it is natural to compare the categorical entropy and the spectral radius on the numerical Grothendieck group. Then we show **the lower-bound**:

**Theorem 7** (K [5]). We have the following:

$$h(\Phi) \geq \log \rho(\mathcal{N}(\Phi)).$$

**Theorem 8.**  $h(\Phi) = \log \rho(\mathcal{N}(\Phi))$  holds for curves (K [3]), varieties with (anti-) ample  $K_X$  (K-Takahashi [4]), abelian surfaces and simple abelian varieties (Yoshioka [8]), where  $\Phi$  is an autoequivalence.

**Remark 9.** In general, **the upper-bound** of Theorem 7 does NOT hold for some autoequivalences of even-dimensional Calabi-Yau hypersurfaces (Y.-W.Fan [2]) and all K3 surfaces (Ouchi [7]). It is an interesting and important problem to characterize such autoequivalences.

## 4. Local entropy

Let  $(R, \mathfrak{m})$  be a Noetherian local ring of dimension  $d$ , and  $\phi : (R, \mathfrak{m}) \rightarrow (R, \mathfrak{m})$  a local homomorphism of finite length. (e.g. finite morphisms are of finite length.)

**Definition 10** (Majidi-Zolbanin-Miasnikov-Szpiro [6]). **The local entropy  $h_{\text{loc}}(\phi)$**  of dynamics  $(R, \mathfrak{m}, \phi)$  is defined as follows.

$$h_{\text{loc}}(\phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \ell_R(\phi^n) \in \mathbb{R}_{\geq 0}, \text{ where } \ell_R(\phi^n) := \ell_R(R/\phi^n(\mathfrak{m})R).$$

**Theorem 11** (MZMS [6]). Suppose  $\text{char } R = p > 0$  and let  $F$  be the Frobenius morphism of  $R$ . Then we have the following:

$$h_{\text{loc}}(F) = d \log p \quad (\Leftrightarrow p = \exp(h_{\text{loc}}(F)/d)).$$

**Theorem 12** (MZMS [6]: An analogue of the Kunz's regularity criterion). In the following properties, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) holds. Moreover, if  $\phi$  is **contracting**, then the converses hold.

- (i)  $R$  is regular.
- (ii)  $\phi : R \rightarrow R$  is flat.
- (iii)  $\ell_R(\phi) = p_\phi^d$ , where  $p_\phi := \exp(h_{\text{loc}}(\phi)/d)$ .

**Proposition 13.** Let  $C$  be a smooth projective curve over  $k = \bar{k}$ ,  $f$  a separable  $k$ -endomorphism with a fixed (closed) point  $x \in C$ . Then  $h_{\text{loc}}(f_x) = \log e_x$  hold, where  $e_x$  is the ramification index of  $f$  at  $x$ .

## 5. Applications to periodic points

We compare the categorical entropy and the local entropy of dynamics on a **periodic point** via semi-conjugate categorical dynamics:

**Theorem 14** (K). Suppose  $f$  has periodic points. Then we have the following:

$$h(\mathbb{L}f^*) \geq \sup \{ h_{\text{loc}}((f^l)_x)/l \mid l \in \mathbb{Z}_{>0}, x : l\text{-periodic point of } f \}.$$

**Corollary 15.** We have the followings.

- (i) If  $f$  has a contracting periodic point, then  $h(\mathbb{L}f^*) > 0$ .
- (ii)  $l$ -periodic point  $x$  of  $f$  satisfying  $\ell_{\mathcal{O}_{X,x}}((f^l)_x)/l > h(\mathbb{L}f^*)$  is singular.

**Corollary 16.** Suppose  $\text{char } k = p > 0$  and let  $F$  be the absolute Frobenius morphism of  $X$ . Then the inequality  $h(\mathbb{L}F^*) \geq d \log p$  holds. Moreover, The equality holds in smooth projective cases.

**Remark 17.** For local rings on periodic (closed) points, the contractivity is equivalent to **the super-attractivity** in the theory of complex dynamics.

**Example 18.** Set  $f(z) := z^m - z^n$  ( $m > n > 1$ ), which is a separable  $k$ -endomorphism of  $\mathbb{P}_k^1$  over  $k = \bar{k}$ . Then,  $z = 0$  is a super-attracting fixed point of  $f$ , and  $h(\mathbb{L}f^*) = \log m > \log n = \log e_0 = h_{\text{loc}}(f_0)$ .

Zariski-density of the set of periodic points of polarized endomorphism due to Fakhruddin is well-known. As an analogy of complex dynamics, it is natural to ask whether the set of contracting (isolated) periodic points is finite.

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